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MELTING AND SOLIDIFICATION WITH INTERNAL RADIATIVE TRANSFER—A GENERALIZED PHASE CHANGE MODEL

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Abstract—A more general melting and solidification model is proposed to account for an interesting phenomenon, namely, the existence of a two-phase zone in which partial phase change can occur. The two-phase zone is attributed to internal melting or solidification (as opposed to surface melting or solidification) induced by internal thermal radiation. The model proposes three physical layers: the liquid and solid layers as in the classical model, and a two-phase mixture (part liquid and part solid) layer in between. It is shown that in the limiting case of opaque media, the thickness of the two-phase layer diminishes and the proposed model reduces to the classical one. A corresponding mathematical formulation is presented for phase change problems with internal radiative transfer, which is used to clarify some inconsistencies in the existing literature. Finally, an exact and two approximate solutions to three simple solidification problems are also presented to illustrate the effect of the two-phase zone on solidification of semi-transparent materials.

NOMENCLATURE

A,	surface area, or coefficient in temperature polynomial;
B,	coefficient in temperature polynomial;
C,	specific heat;
e,	specific internal energy;
$e_{b,\nu}$,	Planck function;
E_n,	exponential integral function of nth kind;
H,	emission from a semi-transparent medium;
J,	radiosity;
K_a,	gray absorption coefficient;
K_c,	conductivity;
K_ν,	spectral absorption coefficient;
n,	mass flux, or index of refraction;
$\mathbf{n}$,	unit vector;
q,	total heat flux;
q_r,	radiative flux;
r,	mass solidification rate per unit volume of the mixture;
S,	location of an interface or propagation front of a given void fraction;
S_1,	first interface;
S_2,	second interface;
t,	time;
T,	temperature;
$\bar{T}$,	mean temperature, $[(T_w^4 + T_m^4)/2]^{1/4}$;
$\mathbf{u}$,	velocity;
V,	volume;
x,	physical coordinate normal to wall.

Greek symbols

α,	solid volume fraction;
α_c,	thermal diffusivity;
β,	a quantity, $2\,K_a n^2 \sigma T_\infty^3/\rho^* C$;
$\gamma_{\nu,j,i}$,	spectral transmittance from region j to other side through an interface;
θ,	dimensionless temperature, T/T_∞;
λ,	latent heat of fusion per unit mass;
$\boldsymbol{\pi}$,	pressure tensor;
ρ,	superficial mass density;
ρ^*,	true mass density;
$\rho_{\nu,j,i}$,	spectral reflectance back to region j by an interface;
σ,	Stefan–Boltzmann constant;
τ,	optical thickness, $K_a x$;
τ_S,	optical thickness, $K_a S_2$;
$\boldsymbol{\tau}$,	stress tensor.

Subscripts

1,	region 1;
2,	region 2;
b,	characteristic thickness similar to boundary layer thickness;
i,	interface;
l,	liquid;
m,	melting point;
o,	first time liquid drops to melting temperature;
s,	solid;
w,	wall;
ν,	frequency;
∞,	far away from wall;
eq,	equivalent.

Superscripts

$^+$,	positive x direction;
$^-$,	negative x direction.

1. INTRODUCTION AND CONCEPT OF INTERNAL PHASE CHANGE

Melting and solidification of materials by heat transfer is of importance in many technical fields and has been a subject of interest for over a century. In the past ten years, considerable attention has been turned from opaque to non-opaque materials, in which the internal thermal radiation can no longer be ignored. Typical examples of applications range from low temperature problems, such as ice melting by solar energy, to high temperature problems, such as crystal growth, the solidification of slag in advanced MHD coal-fired power plants, and the melting or solidification of uranium fuels (a consideration in nuclear reactor safety).

Recent studies to ascertain the effects of internal thermal radiation in non-opaque materials during phase change are numerous [1–10] but exhibit considerable differences in their formulations. Table 1 compares and summarizes those formulations for a simple, classical Stefan problem where the surface of a semi-infinite, semi-transparent liquid, initially at the melting temperature T_m, is suddenly lowered to a constant temperature T_w.

It is seen that while all researchers agree on the differential–integral equation, they disagree on the interfacial condition. Some [1, 2, 10] suggest the inclusion of a net radiative flux in the classical expression of Stefan's interfacial condition, while others [3–5] suggest that two net radiative fluxes be included. On the other hand, a differential term of radiative flux is suggested in ref. [9] while an integral term is suggested in ref. [8]. Finally, ref. [9] simply employed the same condition as that of conduction-controlled solidification. For the same problem, except in the limiting cases of conduction-dominated or extremely opaque materials, the different interfacial conditions would undoubtedly lead to different results. The present study is concerned not so much with the differences in numerical results as with the fundamental question of what the proper condition should be.

All of the existing formulations are based on the premise that there are two distinct phases, one being pure liquid and the other pure solid, separated by an interface located at $X = S_1$. Consequently, they have either missed or overlooked an interesting phenomenon, namely, the possible existence of a two-phase zone between the pure liquid and the pure solid zones when the internal radiation becomes important. To illustrate the concept of 'internal phase change', consider the melting of a semi-transparent, semi-infinite solid at its melting point before exposing to an external thermal radiation heat source. The existing model [1–10], which is the same as the classical, conduction-controlled model, assumes melting from the top and proceeding downward layer by layer, resulting in a pure liquid layer on the top of the pure solid layer as shown in Fig. 1(a). To see if this model is appropriate when the internal thermal radiation is significant, consider a small strip within the solid layer as a control volume and make an energy balance. There

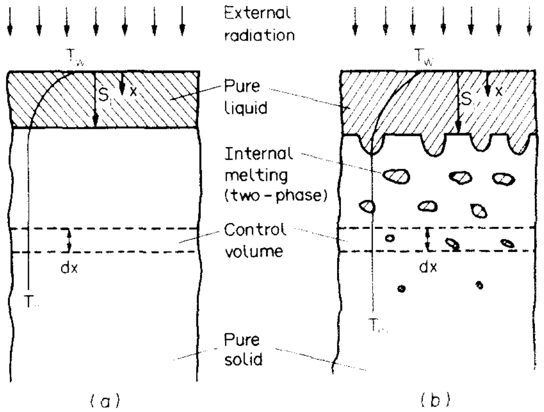
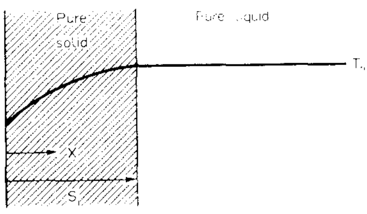


FIG. 1. (a) Existing model. (b) Current model with internal melting (internal phase change).

Table 1. Comparisons of existing formulations for the solidification of a semi-infinite liquid initially at melting temperature

Ref.	Differential equation	Interface condition
[1, 2, 10]	$\rho C \frac{\partial T}{\partial t} = K_c \frac{\partial^2 T}{\partial x^2} - \frac{\partial q_r}{\partial x}$	$K_c \frac{\partial T}{\partial x}\Big\rvert_{S_1^-} - q_r\Big\rvert_{S_1} = \rho^* \lambda \frac{dS_1}{dt}$
[3–5]	same	$K_c \frac{\partial T}{\partial x}\Big\rvert_{S_1^-} - q_r\Big\rvert_{S_1^-} + q_r\Big\rvert_{S_1^+} = \rho^* \lambda \frac{dS_1}{dt}$
[7]	same	$K_c \frac{\partial T}{\partial x}\Big\rvert_{S_1} - \frac{\partial q_r}{\partial x}\Big\rvert_{S_1^-} t \frac{dS_1}{dt} = \rho^* \lambda \frac{dS_1}{dt}$
[8]	same	$K_c \frac{\partial T}{\partial x}\Big\rvert_{S_1^-} - \int_{S_1}^{\infty} \frac{\partial q_r}{\partial x}\, dx = \rho^* \lambda \frac{dS_1}{dt}$
[9]	same	$K_c \frac{\partial T}{\partial x}\Big\rvert_{S_1^-} = \rho^* \lambda \frac{dS_1}{dt}$



are no net energy exchanges in the volume by conduction or convection (density being taken as constant to simplify the discussion here). However, the solid in the small control volume, through its action-at-a-distance characteristics, does absorb part of the thermal radiant energy of the external source. This net gain of radiative energy obviously cannot raise the sensible heat without first melting the solid in the control volume. (Similar to all the studies on this subject [1–10], we consider, without loss of generality, only the pure material with a unique melting temperature and assume no thermal stress or non-equilibrium effects like superheating or supercooling, etc.) Since the amount of energy absorbed is not necessarily exactly equal to the latent heat of melting of the solid in the entire control volume, only partial melting can be expected. Moreover, as the control volume located at the deeper location absorbs less radiative energy, a smaller portion of the solid within the volume is melted. Thus the percentage of solid melted should diminish downward, resulting in a two-phase zone located between the newly formed pure liquid layer on the top and the original solid layer at the bottom as shown in Fig. 1(b).

Similar conclusions can be reached about the possible existence of a two-phase layer (i.e. an internal solidification) in the solidification of a high temperature semi-transparent medium because of the rapid internal radiative cooling.

The existence of a two-phase zone has long been mentioned in the meteorological literature. Dorsey [11] explicitly stated that

> "When a block of ice that is above water is exposed to light, small cavities partly filled with water may be seen to form throughout the body of the ice, provided that the temperature of the ice is not too low. . . . Nevertheless, internal melting occurs, and proves that the interior portions of a mass of ice may be melted by radiant heat which has traversed other portions of the mass without melting them. The consequence is that such a mass, raised to the temperature of 32°F, will have some of its parts liquid and some solid."

Knight [12] also stated that "when an ice crystal is warmed through its volume, especially if by radiation, internal melting occurs." Other references† mentioned the term "internal melting" [7, 9, 13, 14], though with some ambiguity.

From the above, two conclusions can be drawn. First, the internal phase change phenomenon is real and known in the meteorological literature, but somehow has not been addressed in the heat transfer analysis of semi-transparent materials. Secondly, while the detailed microscopic mechanism of internal melting is complex and not well understood (and which is beyond the scope of the present study), the macroscopic end results are obvious, namely, radiant heat sources produce internal melting.

Other than the above observations, an analytical work in nuclear reactor safety also indicates a two-phase layer in the melting of reactor fuels of conduction with an internal heat generation [21]. This further confirms the need to incorporate the two-phase layer in the phase change analysis of semi-transparent materials, since the internal radiation is equivalent to an internal heat source or sink.

2. MATHEMATICAL FORMULATION

2.1. *Model*

A more general phase change model is proposed here which comprises three physical layers [Fig. 1(b)], namely a pure solid layer, a pure liquid layer and a two-phase mixture layer in between. The model proposed is chosen to contrast with the conventional conduction-controlled "two-layer" model, shown in Fig. 1(a). As will be seen later, under the extreme conditions in which the internal radiative transfer is absent or insignificant (such as with opaque metals or transparent materials), the thickness of the two-phase layer diminishes to zero and the proposed "three-layer" model reduces to the conventional "two-layer" model. In this sense, the conventional model can be regarded as a limiting case of the present model, which is applicable to semi-transparent materials as well.

The following mathematical formulation is based on the model proposed for the solidification of a semi-infinite liquid initially at a temperature T_∞, higher than the melting temperature T_m, as shown in Fig. 2. The medium considered is assumed to be semi-transparent, homogeneous and isotropic with constant and identical thermophysical and optical properties in both phases. A unique melting temperature is assumed to exist and the two-phase region is assumed to be in thermal equilibrium at the melting temperature. Since

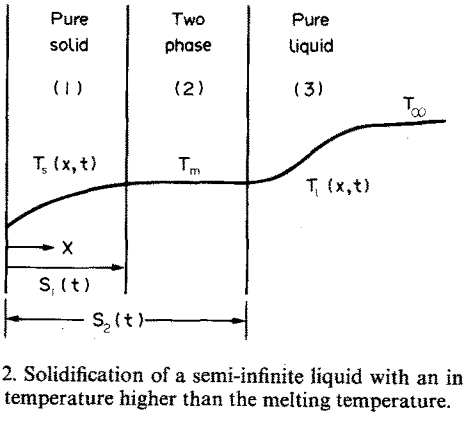


FIG. 2. Solidification of a semi-infinite liquid with an initial temperature higher than the melting temperature.

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† Upon the completion of the paper, it was brought to the authors' attention that a recent work (L. A. Diaz and R. Viskanta, Melting of a slab of semitransparent material by irradiation from an external radiation source AIAA-81-1047, AIAA 16th Thermophysics Conference, June 1981, Palo Alto, CA) observed no internal melting within *n*-octadecane. Unfortunately, the solid *n*-octadecane is opaque (as was so assumed in their paper) so that little internal melting could be expected.

the complex refractive indices of both phases are assumed to be the same, the albedo of scattering is negligible. Finally the change in density during solidification is assumed to be small, such that convective motion can be ignored.

2.2. *Two-phase region*

In order to account for the release of latent heat of solidification by internal solidification, an additional source term, λr, should be included in the energy equation

$$\rho C \frac{\mathrm{D}T}{\mathrm{D}t} - \nabla \cdot K_c \nabla T + \nabla \cdot \mathbf{q}_r = \lambda r. \tag{1}$$

The compressibility and the viscous dissipation terms have been neglected. The transient and conductive terms in equation (1) may be important for eutectic solutions with variable freezing points. With no convection and a constant freezing temperature, equation (1) simplifies to

$$\frac{\partial q_r}{\partial x} = \lambda r. \tag{2}$$

The above is coupled to the continuity equation as the solidification rate is related to the solid volume fraction. Following our recent study [15] on the condensation of a high temperature gas–vapour mixture, the species equation of the solidified phase

$$\frac{\partial \rho_s}{\partial t} = -\nabla \cdot [\rho_s \mathbf{u}_s] + r \tag{3}$$

can be written as

$$\frac{\partial \alpha}{\partial t} = -\nabla \cdot [\alpha \mathbf{u}_s] + \frac{r}{\rho_s^*} \tag{4}$$

in which α is the solidified volume fraction and the superficial solid density, ρ_s, is related to the true density ρ_s^* by $\rho_s = \alpha \rho_s^*$. In the absence of motion ($u_s = 0$) equation (4) reduces to the simple form

$$\frac{\partial \alpha}{\partial t} = \frac{r}{\rho_s^*}. \tag{5}$$

Recalling that the temperature profile in the two-phase region is flat at T_m, the solid volume fraction distribution can then be determined by solving equations (2) and (5). These can be combined as

$$\frac{\partial \alpha}{\partial t} = \frac{1}{\rho_s^* \lambda} \frac{\partial q_r}{\partial x}, \quad t > t_0 \tag{6}$$

subject to the initial condition

$$\alpha = 0 \quad \text{at} \quad t = t_0 \tag{7}$$

where t_0 stands for the time the liquid temperature located at $x = 0$ drops to its melting point. Noting that $\partial q_r / \partial x$ is an integral equation to be supplied from the transfer equation, no other boundary condition needs to be specified.

2.3. *Pure liquid and solid regions*

In the pure liquid region, the formulation is straightforward and can be summarized as

$$\rho_l^* C_l \frac{\partial T}{\partial t} = K_{cl} \frac{\partial^2 T}{\partial x^2} - \frac{\partial q_r}{\partial x}, \quad 0 \leqslant t,\ S_2 \leqslant x \tag{8}$$

with

$$T = T_\infty \quad \text{at} \quad t = 0, \tag{9}$$

$$T = T_m \quad \text{at} \quad x = S_2(t), \tag{10}$$

$$T = T_\infty \quad \text{at} \quad x = \infty \tag{11}$$

where S_2 is the moving front of the second interface between the two-phase and pure liquid regions.

Similarly, for the pure solid phase, the governing equation is

$$\rho_s^* C_s \frac{\partial T}{\partial t} = K_{cs} \frac{\partial^2 T}{\partial x^2} - \frac{\partial q_r}{\partial x}, \quad t_0 \leqslant t,\ x \leqslant S_1 \tag{12}$$

subject to

$$T = T_m \quad \text{at} \quad t = t_0, \tag{13}$$

$$T = T_w \quad \text{at} \quad x = 0, \tag{14}$$

$$T = T_m \quad \text{at} \quad x = S_1(t) \tag{15}$$

where $S_1(t)$ is the first interface between the pure solid and the two-phase region.

2.4. *Interfacial condition at $x = S_1$*

The above equations and conditions are incomplete without further formulating the interfacial conditions at $x = S_1$ and S_2. Special attention is needed for $x = S_1$ as it involves a discontinuity concept due to internal solidification by radiation. As shown in the Appendix, the condition at $x = S_1$ is

$$\left(K_c \frac{\partial T}{\partial x}\right)\bigg|_{S_1^-} - q_r\bigg|_{S_1^-} = \left(K_c \frac{\partial T}{\partial x}\right)\bigg|_{S_1^+} - q_r\bigg|_{S_1^+} + \left(1 - \alpha\bigg|_{S_1^+}\right) \rho_l^* \lambda \left(\frac{\mathrm{d}S_1}{\mathrm{d}t} - \mathbf{u}_l \cdot \mathbf{n}_l\right). \tag{16}$$

The above interface condition is quite general as it allows for the convective motion of the medium should the density change during solidification. In the present problem, no change in density is assumed. Therefore $\rho_l^* = \rho_{S_1}^* = \rho_{S_2}^* = \rho^*$ and $\mathbf{u}_{S_1} = \mathbf{u}_{S_2} = \mathbf{u}_l = 0$. Furthermore, under the assumption of a single melting temperature, $\partial T / \partial x = 0$ at $x = S_1^+$ the interfacial condition at $x = S_1$ reduces to

$$\left(K_c \frac{\partial T}{\partial x}\right)\bigg|_{S_1^-} - q_r\bigg|_{S_1} + q_r\bigg|_{S_1^+} = \left(1 - \alpha\bigg|_{S_1^+}\right) \rho^* \lambda \frac{\mathrm{d}S_1}{\mathrm{d}t} \tag{17a}$$

subject to the initial condition

$$S_1 = 0 \quad \text{at} \quad t = t_0. \tag{17b}$$

Note that the above condition contains a term involving $\alpha(S_1^+)$ which is missing in refs. [1–10]. It should be emphasized that the solid volume fraction on the two-phase side adjacent to the interface, $\alpha(S_1^+)$, is not necessarily unity nor zero; it is a time-dependent unknown to be determined by the solution of the differential equation for the two-phase region [namely, equation (6) subject to equation (7)].

It appears that the differences in the existing interfacial conditions arise from the inconsistent treatment of the radiative flux terms across the interface, namely $q_r(S_1^+)$ and $q_r(S_1^-)$ in equation (17a). For example, it has been stated [3, 4] that if the phases on both sides of the interface have identical opaque and transparent bands then $q_r(S_1^-) = q_r(S_1^+)$. In ref. [7], a term such as $[\partial q_r(S^-)/\partial x]\,\mathrm{d}s$ is proposed. Mathematically, this term represents the first order approximation of $[q_r(S_1^+) - q_r(S_1^-)]$ if the Taylor series expansion is employed. In a later study [9], however, the same authors decided to drop this term. It can be shown that for semi-transparent media on both sides of S_1, the difference between $q_r(S_1^-)$ and $q_r(S_1^+)$ is of the order of $K_v \Delta S$ where ΔS is the distance between S_1^+ and S_1^-, so that $[q_r(S_1^+) - q_r(S_1^-)] \to 0$ as $\Delta S \to 0$. Therefore, the interfacial condition for semi-transparent media can be further simplified to

$$\left(K_c \frac{\partial T}{\partial x}\right)\bigg|_{S_1^-} = \left(1 - \alpha\bigg|_{S_1^+}\right)\rho^* \lambda \frac{\mathrm{d}S_1}{\mathrm{d}t} \tag{18}$$

The inclusion of this continuity (or cancellation) of the radiative fluxes across an interface should be of no surprise since it was used implicitly in an early paper [22].

2.5. *Limiting cases and clarification of existing interface conditions*

In the extreme limit involving an opaque medium (opaque in the sense of having an infinitely large absorption coefficient rather than a large physical dimension), we have $K_a \to \infty$ and $\mathrm{d}S \to 0$, yielding an indeterminate case. However, it can be resolved if one recalls that

$$q_r = -\frac{4n^2\sigma}{3K_a}\frac{\partial T^4}{\partial x}$$

and

$$\frac{\partial q_r}{\partial x} = -\frac{4n^2\sigma}{3K_a}\frac{\partial^2 T^4}{\partial x^2}$$

for an opaque gray medium. As $K_a \to \infty$, $q_r \to 0$ and $[\partial q_r/\partial x] \to 0$. Using these values in equations (6) and (7) yields the obvious solution that $\alpha = 0$ in the two-phase zone. This means that the two-phase zone diminishes completely and we only have two regions left, the pure liquid and the pure solid layers. Therefore, if the original liquid phase is extremely opaque, and the newly solidified phase is, somehow, semi-transparent, the interfacial condition given by equation (17a) for the solidification of a saturated liquid is

$$K_c \frac{\partial T}{\partial x}\bigg|_{S^-} - q_r\bigg|_{S^-} = \rho^* \lambda \frac{\mathrm{d}S}{\mathrm{d}t} \tag{19a}$$

where the subscript 1 has been dropped, since only one interface remains. If the liquid is superheated initially, then a conduction term on the liquid side should be added

$$\left(K_c \frac{\partial T}{\partial x}\right)_{S^-} - q_r\bigg|_{S^-} = \left(K_c \frac{\partial T}{\partial x}\right)_{S^+} + \rho^* \lambda \frac{\mathrm{d}S}{\mathrm{d}t}. \tag{19b}$$

Similarly, if the original phase is semi-transparent but the solidified phase is extremely opaque, the two-phase region diminishes also. This can be easily explained on simple physical grounds. If the solidified phase is extremely opaque, only the solid interface can exchange radiation with the liquid. Since the interface temperature is no lower than the melting point (and neither is the liquid temperature) no internal solidification occurs. Mathematically, $q_r = 0$ and $\partial q_r/\partial x = 0$ everywhere within the liquid region if the liquid is initially at melting temperature. Thus, by equations (6) and (7), $\alpha = 0$, implying the absence of two-phase zone. If the liquid is superheated initially, $\partial q_r/\partial x < 0$ for a monotonically increasing temperature profile and equations (6) and (7) would yield $\alpha < 0$, a physically impossible situation, which further confirms the absence of a two-phase zone. Therefore, for an initially saturated liquid

$$\left(K_c \frac{\partial T}{\partial x}\right)_{S^-} = \rho^* \lambda \frac{\mathrm{d}S}{\mathrm{d}t} \tag{19c}$$

and

$$\left(K_c \frac{\partial T}{\partial x}\right)_{S^-} = \left(K_c \frac{\partial T}{\partial x}\right)_{S^+} + q_r\bigg|_{S^+} + \rho^* \lambda \frac{\mathrm{d}S}{\mathrm{d}t} \tag{19d}$$

for a liquid that is superheated initially. Without further elaboration, it is sufficient to point out that if both phases are opaque, the interfacial condition reduces to

$$\left(K_c \frac{\partial T}{\partial x}\right)_{S^-} = \left(K_c \frac{\partial T}{\partial x}\right)_{S^+} + \rho^* \lambda \frac{\mathrm{d}S}{\mathrm{d}t} \tag{19e}$$

which is identical to the conventional heat conduction condition.

Under the limiting cases examined above, all the q_r terms in equations (19a)–(19d) or Table 1 are in fact numerically equal to zero and should be dropped. The end results are the same as if we have examined the limiting cases of equation (18) rather than equation (17a). This can be readily proved by examining the interface between a semi-transparent medium and an absolutely opaque medium. By applying Fresnel relationships across the interface one will find a perfect reflection on the semi-transparent side and no emission from the opaque to the semi-transparent medium, implying no net fluxes on either side of the interface. These redundant q_r terms are nevertheless kept in equations (19a)–(19d) for ease of comparison with those in Table 1 since the latter, except one, contain some forms of q_r.

The above discussion reveals that the existing formulations [1–5, 9, 10] in Table 1 are applicable in the limiting case when the medium is opaque on one or both sides of the interface with a negligible two-phase zone. The present formulation, however, is preferable when applied to the general case where the semi-transparent medium remains semi-transparent after phase change.

2.6. *Interfacial condition at $x = S_2$*

Following the derivation of the condition at $x = S_1$, the other interfacial condition at $x = S_2$ in Fig. 2 can be obtained analogously to equation (A11) as

$$\mathbf{q}_3 \cdot \mathbf{n}_3 + \mathbf{q}_2 \cdot \mathbf{n}_2 = \alpha|_{S_2} \, \rho^*_{S_2} \lambda (\mathbf{u}_{S_2} - \mathbf{u}_i) \cdot \mathbf{n}_2 \tag{20}$$

in which the unit vector $\mathbf{n}_3$ points in the positive x direction while $\mathbf{n}_2$ points in the opposite direction. Assuming no change in density during the phase change and taking into consideration the continuity of the radiative flux, it simplifies to

$$K_{c2} \left.\frac{\partial T}{\partial x}\right|_{S_2^-} - K_{c3} \left.\frac{\partial T}{\partial x}\right|_{S_2^+} = \alpha\Big|_{S_2} \rho^* \lambda \frac{\mathrm{d}S_2}{\mathrm{d}t} \tag{21a}$$

subject to the initial condition

$$S_2 = 0 \quad \text{at} \quad t = t_0. \tag{21b}$$

In the problem of interest, a single melting temperature is assumed. Thus,

$$\left.\left(\frac{\partial T}{\partial x}\right)\right|_{S_2} = 0.$$

Now, for the solidification of a liquid initially at a higher temperature than the melting point, $\mathrm{d}S_2/\mathrm{d}t > 0$ and equations (21a) and (21b) would predict a negative temperature gradient at $x = S_2^+$ if $\alpha(S_2^-)$ has other than zero or negative values. This is, of course, physically impossible, as are negative α's. Therefore, we conclude that the interfacial condition at $x = S_2^-$ will have to be

$$\alpha = 0 \quad \text{at} \quad x = S_2 \tag{22}$$

which implies

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = S_2^+. \tag{23}$$

The concept of $\alpha = 0$ at $x = S_2$ and a discontinuous value of α at $x = S_1$ is in fact similar to what happens in the solidification of an opaque binary eutectic alloy [18–20], namely, a zero value of the solid fraction at the liquidus front and a finite but discontinuous value at the solidus front.

Mathematically, one has to solve for the solid volume fraction profile, $\alpha = \alpha(x, t)$, for the two phase region. Then by setting the resulting solution equal to zero at $x = S_2(t)$, i.e. $\alpha[S_2(t), t] = 0$, the growth of the second interface, $S_2 = S_2(t)$, is determined implicitly.

2.7. *Interfacial conditions and governing differential equations for melting problems*

For completeness, the analogous formulations for melting problems are briefly summarized here. Consider a semi-infinite solid initially at a uniform temperature T_∞ below the melting point, T_m, when its surface temperature is raised to $T_w(> T_m > T_\infty)$. The formation of three regions like those in Fig. 2 can be expected, except region 1 becomes a pure liquid while region 3 remains as the original solid. The governing differential equations for each region remain unchanged. As for the interfacial conditions, a discontinuity in the solid volume fraction exists at the liquid–two-phase interface, but not at the second interface. They can be readily obtained as

$$-K_{c,1} \left.\frac{\partial T}{\partial x}\right|_{S_1} = \alpha\Big|_{S_1^+} \rho^* \lambda \frac{\mathrm{d}S_1}{\mathrm{d}t} \tag{24}$$

and

$$\alpha|_{S_2^-} = 1 \tag{25}$$

at $x = S_1$ and S_2, respectively. For opaque cases, results similar to equations (19a)–(d) can be derived and will not be repeated here.

3. ANALYTICAL SOLUTIONS

To elucidate the concept of the internal phase change, some highly simplified solidification problems are to be examined and solved using the formulation presented here.

Consider a gray semi-transparent, semi-infinite liquid bounded by a black surface. The medium is assumed to have a distinct melting point and the non-equilibrium effects, such as subcooling etc., are neglected. The solid and liquid in the two-phase mixture are assumed to thermally equilibrate instantaneously. Thermophysical and radiation properties are assumed to be constant and identical in both liquid and solidified phases. Since the refractive indices remain unchanged during the phase change, the albedo of scattering within the medium can be neglected.

3.1. *Solidification of a liquid at melting point Radiation-controlled solidification*

The first specific problem examined is the solidification of a pure liquid, initially at its melting point, T_m, when its surface temperature is suddenly lowered to $T_w < T_m$. The conduction–radiation parameter ($N = K_c K_a / 4n^2 \sigma T_m^3$) is assumed to be small enough that the solidification is solely controlled by thermal radiation. This is not as restrictive as it sounds since it accurately represents the solidification of high temperature liquids. The other extreme of $N \to \infty$ represents the well known conduction problem which has been thoroughly examined in the literature. In as much as the primary attention is focussed on the characteristics of the internal solidification, we would further restrict ourselves in this problem and the next to the solidification process up to the time when the liquid at $s = 0+$ reaches 100% solidification by radiative cooling. Since the temperature remains unchanged throughout the medium, the governing equations in the two-phase region are given by equations (6) and (7) with

$$\frac{\partial q_r}{\partial x} = 2K_a n^2 \sigma (T_m^4 - T_w^4) E_2(\tau) \tag{26}$$

where τ and x are measured from (i.e. normal to) the wall. Clearly, the heat transfer to the wall is $q_r = n^2\sigma(T_m^4 - T_w^4)$ from equation (26), and the local solid volume fraction can be easily solved to yield

$$\alpha(x, t) = E_2(\tau) t^* \tag{27}$$

where the dimensionless time is defined by

$$t^* = 2K_a n^2 \sigma (T_m^4 - T_w^4) t / \rho^* \lambda.$$

Similarly, the internal local solidification rate from equation (2) is

$$r = \frac{1}{\lambda}\frac{\partial q_r}{\partial x} = \frac{2K_a n^2 \sigma}{\lambda}(T_m^4 - T_w^4) E_2(\tau) \tag{28}$$

from which the total solidification rate throughout the medium can be found,

$$r_{\text{total}} = \int_0^\infty r \, dx = n^2 \sigma (T_m^4 - T_w^4)/\lambda \tag{29}$$

as is expected.

The local volume fraction and the local internal solidification rate are displayed in Fig. 3. It is seen that they decay exponentially with increasing distance from the wall and are linearly proportional to the fourth power of temperature, two unique characteristics that arise when radiative transfer is dominant. Furthermore, the internal solidification rate is independent of time because of the constant and uniform temperature throughout the medium, so the solid fraction at a given location increases linearly with increasing time.

The total solidification rate can be utilized to find the equivalent 100% pure solid layer thickness when all the solidified fractions are packed together,

$$S_{\text{eq}} = \frac{1}{K_a}\int_0^\infty \alpha \, d\tau = \frac{1}{\rho^*}\int_0^t r_{\text{total}} \, dt = \frac{n^2\sigma(T_m^4 - T_w^4)t}{\lambda\rho^*}, \tag{30}$$

which illustrates a linear growth in contrast to the conventional square root of time growth in conduction-controlled solidification.

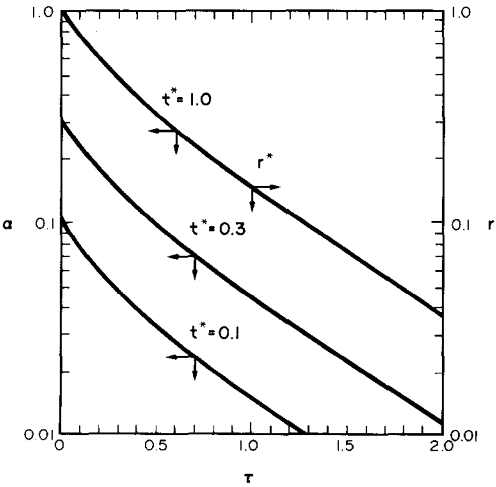

FIG. 3. Local solidification rate and solid void fraction of a liquid at freezing point by radiative cooling ($t^* \equiv 2K_a n^2\sigma(T_m^4 - T_w^4)t/\rho^*\lambda$ and $r^* \equiv \lambda r / 2K_a n^2 \sigma (T_m^4 - T_w^4)$).

Finally, another observation can be made regarding the characteristic thickness of the two-phase layer. In accordance with equation (27), the internal solidification penetrates to an infinite extent because $\alpha \to 0$ only when $x \to \infty$. However, if a two-phase layer thickness is alternatively defined as one whose solid fraction at the outer edge of the layer is equal to 1% of that at the innermost edge, i.e.

$$\alpha(x_b, t)/\alpha(0+, t) = 0.01, \tag{31}$$

then from equation (27) we find $E_2(\tau_b) = 0.01$, which yields the characteristic optical thickness of the layer as

$$\tau_b = 3. \tag{32}$$

Therefore, the less opaque the medium, the thicker the two-phase layer and vice versa.

It is seen that the exact solution has been obtained for the above simple problem. In the following, two slightly more complicated problems are considered. Exact solutions appear to be unlikely; only approximate solutions will be attempted.

3.2. *Solidification of a liquid above melting point—Radiation-controlled solidification*

The problem considered is identical to the above except that the fluid is initially at a temperature, T_∞, higher than the melting point. In the pure liquid region, equation (8) is simplified to

$$\rho^* C \frac{\partial T}{\partial t} = -\frac{\partial q_r}{\partial x}, \quad 0 < t \tag{33}$$

with the initial condition given by equation (9) and

$$-\frac{\partial q_r}{\partial \tau} = 2n^2\sigma T_w^4 E_2(\tau) + 2n^2 \int_0^\infty T^4(\tau') \times E_1(|\tau - \tau'|)\, d\tau' - 4n^2\sigma T^4(\tau). \tag{34}$$

The nonlinear integral form of the radiative flux makes the closed form solution unlikely. However, this equation can be greatly simplified using the fact that $E_1(|\tau - \tau'|)$ behaves as a delta function. As $\tau' \to \tau$, $E_1 \to \infty$ and as $|\tau - \tau'|$ increases, E_1 diminishes rapidly. As a result, E_1 behaves like a delta function and equation (35) can be approximated by

$$-\frac{dq_r}{d\tau} = 2n^2\sigma T_w^4 E_2(\tau) + 2n^2\sigma T^4(\tau)\int_0^\infty E_1(|\tau - \tau'|) \times d\tau' - 4n^2 T^4(\tau) = 2E_2(\tau) n^2 \sigma [T_w^4 - T^4(\tau)].$$

This approximation has been used and found to be satisfactory in our previous analyses [15, 23, 24]. In the present problem of interest, $T^4 \gg T_w^4$, thus

$$\frac{\mathrm{d}q_r}{\mathrm{d}\tau} \simeq 2n^2\sigma T^4(\tau)E_2(\tau). \tag{35}$$

Employing the above approximation in equation (33), the transient temperature profile in the pure liquid region can be found as

$$\theta^3 = 1/[1+3\beta E_2(\tau)t] \tag{36}$$

where

$$\theta = T/T_\infty$$

and

$$\beta = 2K_a n^2\sigma T_\infty^3/\rho^* C.$$

Internal solidification starts as soon as the temperature drops to the melting temperature, T_m. By setting $\theta = \theta_m (\equiv T_m/T_\infty)$ in equation (36) yields

$$E_2(\tau_{S_2}) = (\theta_m^{-3}-1)/3\beta t \tag{37}$$

from which the growth of the two-phase layer thickness, $\tau_{S_2} \equiv K_a S_2$, can be found implicitly as a function of time.

Within the delta function approximation, it is noted that the divergence of the radiative flux given by equation (35) depends only on the local temperature and, naturally, the temperature profile given by equation (36) remains valid in the liquid layer (i.e. $\tau > \tau_s$) even after the appearance of the two-phase layer.

Turning now to the two-phase region which emerges after $t = t_0$ defined by

$$t_0 = (\theta_m^{-3}-1)/3\beta E_2(\tau_s)|_{\tau_s=0} = (\theta_m^{-3}-1)/3\beta. \tag{38}$$

The temperature within the region should be isothermal at $T = T_m$. The governing equations for α are again given by equations (6) and (7). Similar to equation (27), the solution is

$$\alpha(\tau, t) = \frac{2K_a n^2\sigma(T_m^4 - T_w^4)}{\rho^*\lambda} E_2(\tau)(t-t_0). \tag{39}$$

Once the solid fraction and temperature distributions are known, other quantities of interest can be evaluated. For example, in the absence of conduction and convection, the radiative flux represents the total wall heat flux,

$$-q_{rw}(t) = 2n^2\sigma T_\infty^4 \int_0^\infty \theta^4(\tau, t)E_2(\tau)\,\mathrm{d}\tau. \tag{40}$$

Prior to solidification, i.e. $t < t_0$, θ is given by equation (36). Thus

$$-\frac{q_{rw}(t)}{2n^2\sigma T_\infty^4} = \int_0^\infty \frac{E_2(\tau)}{[1+3\beta t E_2(\tau)]^{4/3}}\,\mathrm{d}\tau. \tag{41}$$

After solidification begins (at $t > t_0$) two distinct regions appear,

$$-\frac{q_{rw}(t)}{2n^2\sigma T_\infty^4} = \theta_m^4[0.5 - E_3(\tau_s)] + \int_{\tau_s(t)}^\infty \frac{E_2(\tau)}{[1+3\beta t E_2(\tau)]^{4/3}}\,\mathrm{d}\tau \tag{42}$$

where the first term on the RHS can be attributed to the internal solidification region while the second refers to the pure liquid region. The above integrations can be made numerically in a straightforward manner.

If the internal solidification is neglected, the wall flux given by equation (41) would be applicable not only for $t < t_0$ but for $t > t_0$ as well. Then a comparison with equation (42) shows that the heat flux to the wall is underestimated by the amount

$$\frac{\Delta q_{rw}(t)}{2n^2\sigma T_\infty^4} = \int_0^{\tau_s} [\theta_m^4 - \theta^4(\tau, t)]E_2(\tau)\,\mathrm{d}\tau$$

where θ is described by equation (36). As time increases ($t > t_0$ of course), θ^4 falls rapidly below the value of θ_m^4 and, therefore, the error in the wall flux rises sharply. This can be explained on physical grounds. In the presence of the internal solidification, by virtue of the high latent heat of fusion relative to sensible heat effects, a temperature will be maintained at high levels (i.e. sustained at the melting point), which in turn is supportive of the radiative heat transport mechanism, resulting in a great enhancement of heat transfer to the surrounding wall. Such an effective heat transfer mechanism is basically similar to the fog formation process we discussed recently in connection with the condensation of a hot vapor mixture [15].

3.3. *Solidification of a liquid at melting point—Combined conduction and radiation*

As a final example, consider a more difficult problem to include the conduction heat transfer in the first problem. In addition to the two-phase region, a solidified layer composed of 100% solid is induced instantaneously by conduction near the surface at $x = 0$. To keep the problem amenable without having to resort to complicated numerical methods, we restrict ourselves to using the optically thin approximation for the solid layer and linearizing T^4 terms in the radiative flux expression.

First, consider the two-phase region ($x > S_1$). As long as the solid layer is optically thin, the divergence of radiative flux is identical to equation (26) in the first problem. Since the differential equation and the initial condition are also identical, the solid fraction distribution and others obtained in problem 1 are still valid here except for the equivalent thickness of the solid layer which should now be

$$S_{eq} = S_1(t) + \frac{1}{K_a}\int_{\tau_{S_1}}^\infty \alpha\,\mathrm{d}\tau$$
$$= S_1(t) + \frac{2n^2\sigma(T_m^4 - T_w^4)}{\rho^*\lambda} E_3(\tau_{S_1})t \tag{43}$$

where $S_1(t)$ is the solid layer thickness to be determined next.

In the solid layer, the governing equations are given by equations (12)–(15) except $t_0 = 0$ and the radiative term can be linearized as

$$-\frac{\partial q_r}{\partial \tau} = 2n^2\sigma(T_w^4 + T_m^4 - 2T^4) = 16n^2\sigma\bar{T}^3(T - \bar{T}) \tag{44}$$

with

$$\bar{T}^4 \equiv \tfrac{1}{2}(T_w^4 + T_m^4). \tag{45}$$

They should be solved simultaneously with the interfacial condition, obtained by combining equations (18) and (27),

$$K_c \frac{\partial T}{\partial x}\bigg|_{S_1^-} = [\rho^*\lambda + 2K_a n^2\sigma(T_w^4 - T_m^4)E_2(\tau_{S_1})t]\frac{dS_1}{dt} \tag{46}$$

with

$$S_1 = 0 \quad \text{at} \quad t = 0. \tag{47}$$

The above set of equations with a moving boundary condition can be solved by the integral technique, the technique used extensively in studying the conduction and radiation solidification problems [1, 25]. We assumed a second degree polynomial,

$$T_m - T = A(x - S_1) + B(x - S_1)^2, \tag{48}$$

with three unknown time dependent constants A, B and S_1 to be determined by the integral method. Though somewhat tedious and lengthy, the solution procedure is generally straightforward. Only the final results are given below:

$$A = [\delta - (\delta^2 + 4\eta)^{1/2}]/2 \tag{49}$$

with

$$\begin{aligned} \delta &= 2f/S_1, \\ f &= [1 + C_1E_2(\tau_{S_1})t]\lambda/C, \\ C_1 &= 2K_a n^2\sigma(T_w^4 - T_m^4)/\rho^*\lambda, \\ \eta &= f(2T_m - 2T_w + C_2S_1^2)/S_1^2, \\ C_s &= 16K_a n^2\sigma\bar{T}^3(T_m - \bar{T})/K_c, \\ \tau_{S_1} &= K_aS_1 \end{aligned}$$

and

$$B = (T_m - T_w + AS_1)/S_1^2. \tag{50}$$

The solid layer thickness is to be determined by solving

$$\begin{aligned} \frac{dS_1}{dt}\left[\frac{AS_1}{3} - \frac{T_m - T_w}{3} - \frac{S_1^2}{6}g - f\right] \\ = -\frac{S_1^2}{6}h - \alpha_c\left[A + \frac{2(T_m - T_w)}{S_1}\right] - \frac{16K_a n^2\bar{T}^3}{\rho^*C} \\ \left\{\left[\frac{2(T_m - \bar{T}) + (T_w - \bar{T})}{3}\right]S_1 + \frac{A}{6}S_1^2\right\} \end{aligned} \tag{51}$$

with the initial condition equation (47). In the above equation,

$$\begin{aligned} g = \left(1 - \frac{\delta}{2P}\right)\left[\frac{f}{S_1^2} + \frac{C_1\lambda K_a}{S_1C}tE_1(\tau_{S_1})\right] \\ -\left[\frac{4f}{S_1^3}(T_m - T_w) + \frac{C_1\lambda K_a}{C}\right. \\ \left.\times\left(\frac{2T_m - 2T_w}{S_1^2} + C_2\right)tE_1(\tau_{S_1})\right]\bigg/2P, \end{aligned}$$

$$P = [\delta^2/4 + \eta]^{1/2},$$

$$h = \frac{C_1\lambda}{2C}E_2(\tau_{S_1}) \times\left[\frac{2}{S_1}\left(1 - \frac{\delta}{2P}\right) - \left(\frac{2T_m - 2T_w}{PS_1^2} + \frac{C_2}{P}\right)\right].$$

While the closed form solution appears unlikely, equation (51) can be easily solved numerically by any standard method, such as a Runge–Kutta or Gear method. In the absence of radiation, setting $K_a = 0$ in equation (51) yields

$$S_1\frac{dS_1}{dt} = \frac{W^2}{2}$$

which yields

$$S_1 = W(t)^{1/2}$$

with

$$W = 2(3\alpha_C)^{1/2}\left[\frac{1 - (1 + \mu)^{1/2} + \mu}{5 + (1 + \mu)^{1/2} + \mu}\right]^{1/2},$$

$$\mu = 2(T_m - T_w)C/\lambda,$$

in agreement with ref. [25].

As a numerical example, Fig. 4 illustrates the solid fraction distribution throughout the medium at various times. The region in which $\alpha = 1$ is the solid

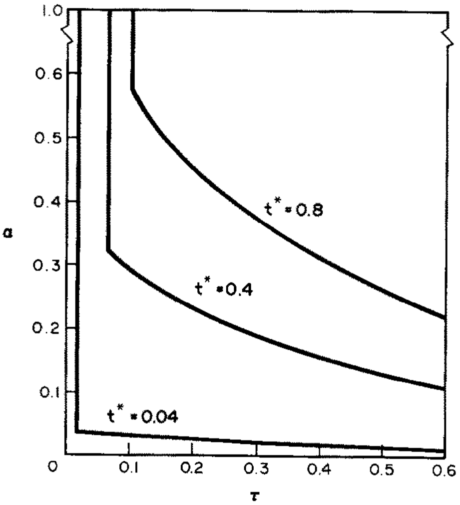

FIG. 4. Solidified void fraction profile. $T_w = 1000$ K, $T_m = 1111$ K, $\alpha_c = 0.0024\ \text{m}^2\,\text{h}^{-1}$, $\rho = 2403\ \text{kg}\,\text{m}^{-3}$, $K_c = 1.339\ \text{W}\,\text{m}^{-1}\,\text{K}^{-1}$, $n = 1.5$, $\lambda = 4.652 \times 10^5\ \text{J}\,\text{kg}^{-1}$, $C = 0.8368\ \text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$, and $K_a = 3.937\ \text{cm}^{-1}$.

layer which grows with time. Clearly α suffers a discontinuity at the interface, beyond which α reduces exponentially in the two-phase region. It is important to note that the two-phase region is considerably thicker than the pure solid layer. Physically what happens to the solid fraction distribution is as follows. Initially at $t = 0$, conduction heat transfer will trigger an immediate solidification of the liquid at $x = 0$ layer by layer, while the radiative heat transfer will induce some solidification away from $x = 0$ inside the liquid. In other words, conduction heat transfer is primarily responsible for the surface solidification which leads to the formation and growth of a pure solid layer ($\alpha = 1$) whereas the radiative cooling is responsible for the internal solidification which produces a non-zero solid fraction ahead of the pure solid layer.

An alternate but interesting plot is shown in Fig. 5, in which the locations of the propagation fronts of various fractions in the medium are displayed. The curve $\alpha = 100\%$ represents the interface front of the pure solid layer which is slightly ahead of the layer without considering radiation. At a given time, say, $t = 0.4$ h, the penetration of the 30% and 10% solidification wavefronts are, respectively, at least four and ten times ahead of that of the 100% front. Also exhibited in the figure is the equivalent solidified thickness that would result if all the solidified medium is packed together. It is seen that if the radiation heat transfer is ignored, it would take four times longer to produce an equivalent solidification layer thickness of 0.01 m and five times longer to produce a 0.02 m solidified layer. It also should be noted that the equivalent thickness is substantially thicker than the 100% curve, indicating that the solidification process has been greatly augmented by the internal solidification and that the internal solidification is in fact much more significant than the surface solidification.

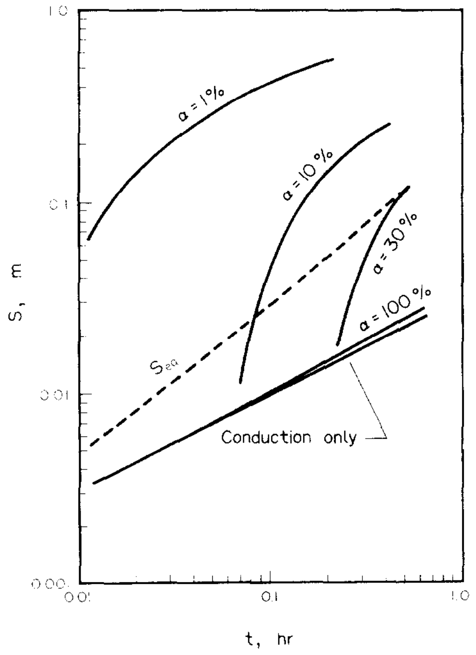

Fig. 5. Propagation fronts of various void fractions (same data as Fig. 4).

The above simple solution based on the integral technique in fact agrees reasonably well with the more sophisticated solution without linearization. The latter can be found elsewhere [26].

4. CONCLUSIONS

The present study has suggested a potentially more general phase-change model than the current one. Governing differential equations and interfacial conditions have been reformulated for a class of melting and solidification problems involving internal thermal radiation. They have been found useful in clarifying certain inconsistencies in the existing formulations of the interfacial condition. When one of the two phases of a medium is absolutely opaque, the internal phase change diminishes and the present model is shown to reduce to the classical model which ignores the two-phase region.

Analytical solutions, one exact and two approximate, to three specific problems have been obtained to illustrate the unique characteristics of internal solidification as opposed to conventional surface solidification. The results show that the existence of the two-phase layer is not only fundamentally important but also has great significance in practical applications as the layer thickness can be considerably thicker and the mass solidified within can be more substantial than that of the pure solid layer.

Finally, it should be pointed out that while the present formulation is fairly general, it is based on some idealizations which could be gradually removed later. In particular, it would be of interest to remove the assumption that the absorption coefficient is constant by using different absorption coefficients for different phases, or by incorporating into analyses the dependence of both absorption and scattering coefficients on evolution of internal melting or solidification. It is noted that the basic formulation [namely, the energy equation (1), the continuity equation (4) and the interface condition, equation (18)] is applicable to the case that considers scattering in the two-phase region. The equilibrium assumption should be relaxed to examine the non-equilibrium effects, such as supercooling, etc. Extensions to allow for eutectic media which have no single melting temperature and to include buoyancy effects in the two-phase region due to the density difference are a few more examples that remain to be studied.

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APPENDIX

INTERFACIAL CONDITION AT S_1

Consider a material volume $V(t)$ with an external material surface $A(t)$ as shown in Fig. 6. The volume $V(t)$ is composed of two volumes $V_1(t)$ and $V_2(t)$ divided by an internal surface $A_i(t)$. The subscripts 1 and 2 represent the pure solid and the two-phase regions, respectively. Similarly, the external surface of $V(t)$ is the sum of the two surfaces $A_1(t)$ and $A_2(t)$, embedded in the two regions. As the internal surface $A_i(t)$ moves with an arbitrary velocity $u_i(t)$, let the quantity F per unit mass of the material suffer a discontinuity across the internal surface with F_1 and F_2 as the quantities on each of the two sides. If $\bar{\phi}$ is the efflux of the quantity F through the bounding surface $A(t)$, and $\dot{F}$ is the rate of generation of the quantity per unit mass at each point throughout the volume $V(t)$, then the general integral balance for the quantity F can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho F\,\mathrm{d}V = -\iint_{A(t)} \mathbf{n}\cdot\boldsymbol{\phi}\,\mathrm{d}A + \iiint_{V(t)} \rho\dot{F}\,\mathrm{d}V \tag{A1}$$

where $\mathbf{n}$ is the unit normal vector directed outwardly with respect to the boundary surface $A(t)$. Application of the generalized Reynolds transport theorem [16, 17] to the term on the LHS of equation (A1) and rearrangement results in

$$\iiint_{V(t)} \frac{\partial(\rho F)}{\partial t}\,\mathrm{d}V + \iint_{A(t)} \mathbf{n}\cdot(\rho F\mathbf{u}+\boldsymbol{\phi})\,\mathrm{d}A + \iint_{A_i(t)} \mathbf{n}_i\cdot(\rho_1 F_1 - \rho_2 F_2)\mathbf{u}_i\,\mathrm{d}A = \iiint_{V(t)} \rho\dot{F}\,\mathrm{d}V \tag{A2}$$

where $\mathbf{n}_i$ is the unit vector normal to $A_i(t)$ in the direction from V_1 to V_2. We let the areas $A_1(t)$ and $A_2(t)$ shrink down to $A_i(t)$, so the total volume $V(t) \to 0$ while the area of the interface, $A_i(t)$, remains finite. The volume integrals vanish and $\mathbf{n}_1 \to -\mathbf{n}_i$, $\mathbf{n}_2 \to \mathbf{n}_i$ in the limit and equation (A2) becomes

$$\iint_{A_i(t)} \{\mathbf{n}_1\cdot[\rho_1 F_1(\mathbf{u}_1-\mathbf{u}_i)+\boldsymbol{\phi}_1]+\mathbf{n}_2\cdot[\rho_2 F_2(\mathbf{u}_2-\mathbf{u}_i)+\boldsymbol{\phi}_2]\}\,\mathrm{d}A = 0.$$

Since this relation holds for any area $A_i(t)$, the integrand must vanish identically to yield

$$\mathbf{n}_1\cdot[\rho_1 F_1(\mathbf{u}_1-\mathbf{u}_i)+\boldsymbol{\phi}_1]+\mathbf{n}_2\cdot[\rho_2 F_2(\mathbf{u}_2-\mathbf{u}_i)+\boldsymbol{\phi}_2] = 0 \tag{A3}$$

which is the general balance equation at the interface.

From equation (A3) the balances concerning mass, momentum and energy can be obtained with proper

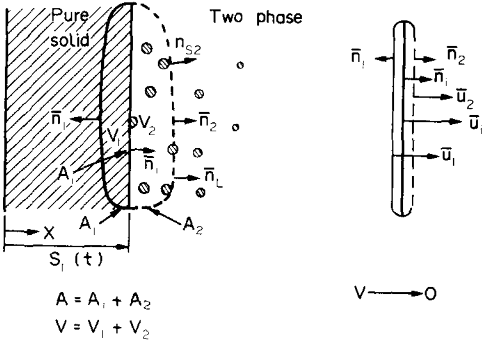

FIG. 6. Control volume on interface.

identification of the quantity F and the efflux $\boldsymbol{\phi}$. The mass balance is obtained with $F = 1$ and $\boldsymbol{\phi} = 0$, viz.

$$\rho_1(\mathbf{u}_1 - \mathbf{u}_i)\cdot\mathbf{n}_1 + \rho_2(\mathbf{u}_2 - \mathbf{u}_i)\cdot\mathbf{n}_2 = 0. \qquad \text{(A4)}$$

To obtain the energy balance we set

$$F = e + \frac{u^2}{2},$$

$$\boldsymbol{\phi} = \mathbf{q} - \boldsymbol{\pi}\cdot\mathbf{u}$$

where $\mathbf{q}$ is the net flux which consists of conductive and radiative fluxes, e is the specific internal energy, and $\boldsymbol{\pi}$ is the pressure tensor. With these identification for F and $\boldsymbol{\phi}$ the energy balance becomes

$$\mathbf{n}_1\cdot\left[\rho_1\left(h_1 + \frac{u_1^2}{2}\right)(\mathbf{u}_1 - \mathbf{u}_i) + P_1\mathbf{u}_i - \boldsymbol{\tau}_1\cdot\mathbf{u}_1 + \mathbf{q}_1\right]$$

$$+\mathbf{n}_2\cdot\left[\rho_2\left(h_2 + \frac{u_2^2}{2}\right)(\mathbf{u}_2 - \mathbf{u}_i) + P_2\mathbf{u}_i - \boldsymbol{\tau}_2\cdot\mathbf{u}_2 + \mathbf{q}_2\right] = 0 \qquad \text{(A5)}$$

where the relationships $\pi = -p\boldsymbol{\delta} + \boldsymbol{\tau}$ and $h = e + p/\rho$ have been used. Usually, in comparison with the thermal energy exchange, the mechanical energy exchange and the work done by viscous stresses can be neglected to yield

$$\rho_1 h_1(\mathbf{u}_1 - \mathbf{u}_i)\cdot\mathbf{n}_1 + \rho_2 h_2(\mathbf{u}_2 - \mathbf{u}_i)\cdot\mathbf{n}_2 + \mathbf{q}_1\cdot\mathbf{n}_1 + \mathbf{q}_2\cdot\mathbf{n}_2 = 0. \qquad \text{(A6)}$$

Since region 1 contains only the pure solid, we have

$$\rho_1(\mathbf{u}_1 - \mathbf{u}_i)\cdot\mathbf{n}_1 = \rho_s^*(\mathbf{u}_{s1} - \mathbf{u}_i)\cdot\mathbf{n}_1, \qquad \text{(A7)}$$

$$\rho_1 h_1(\mathbf{u}_1 - \mathbf{u}_i)\cdot\mathbf{n}_1 = \rho_s^* h_s(\mathbf{u}_{s1} - \mathbf{u}_i)\cdot\mathbf{n}_1. \qquad \text{(A8)}$$

On the other hand, region 2 contains a two-phase mixture. Thus,

$$\rho_2(\mathbf{u}_2 - \mathbf{u}_i)\cdot\mathbf{n}_1 = (1 - \alpha|_{S_1^+})\rho_l^*(\mathbf{u}_l - \mathbf{u}_i)\cdot\mathbf{n}_l + \alpha|_{S_1^+}\rho_s(\mathbf{u}_{s2} - \mathbf{u}_i)\cdot\mathbf{n}_{s2} \qquad \text{(A9)}$$

and

$$\rho_2 h_2(\mathbf{u}_2 - \mathbf{u}_i)\cdot\mathbf{n}_2 = (1 - \alpha|_{S_1^+})\rho_l^* h_l^*(\mathbf{u}_l - \mathbf{u}_i)\cdot\mathbf{n}_l + \alpha|_{S_1^+}\rho_{s2}^* h_s^*(\mathbf{u}_{s2} - \mathbf{u}_i)\cdot\mathbf{n}_{s2}. \qquad \text{(A10)}$$

The above four equations are substituted into equations (A4) and (A6), and the resulting equations can be combined by eliminating the term $\rho_{s1}^*(\mathbf{u}_{s1} - \mathbf{u}_i)\cdot\mathbf{n}_i$ between them. After considerable manipulations, we obtain the interfacial condition at $x = S_1$,

$$\mathbf{q}_1\cdot\mathbf{n}_1 + \mathbf{q}_2\cdot\mathbf{n}_2 = (1 - \alpha|_{S_1^+})\rho_l^*\lambda(\mathbf{u}_i - \mathbf{u}_l)\cdot\mathbf{n}_l. \qquad \text{(A11)}$$

Noting that

$$\mathbf{u}_i\cdot\mathbf{n}_l = \mathrm{d}S_1/\mathrm{d}t,$$

$$\mathbf{q}_1\cdot\mathbf{n}_1 = \left(K_c\frac{\partial T}{\partial x}\right)\bigg|_{S_1^-} - q_r\bigg|_{S_1^-}$$

and

$$\mathbf{q}_2\cdot\mathbf{n}_2 = -\left(K_c\frac{\partial T}{\partial x}\right)\bigg|_{S_1^+} + q_r\bigg|_{S_1^+},$$

equation (A11) can be written finally as equation (16) in the main text.

FUSION ET SOLIFICATION AVEC TRANSFERT RADIATIF INTERNE—UN MODELE DE CHANGEMENT DE PHASE

Résumé—On propose un modèle général de fusion et de solification qui tient compte d'un phénomène intéressant qui est l'existence d'une zone diphasique dans laquelle un changement partiel de phase peut s'opérer. La zone diphasique est attribuée à la fusion ou à la solification interne (en opposition avec fusion ou solidification superficielle) induite par un rayonnement thermique. Le modèle propose trois couches physiques: les couches liquide et solide comme dans le modèle classique et entre elles une couche diphasique (partiellement liquide et solide). On montre que dans le cas limite d'un milieu opaque, l'épaisseur de la couche diphasique diminue et le modèle proposé se réduit au cas classique. Une formulation mathématique est présentée pour les problèmes de changement de phase avec transfert radiatif interne; elle clarifie quelques points inconsistants de la littérature. Une solution exacte et deux autres approchées pour trois problèmes simples de solidification sont présentées pour illustrer l'effet de la zone diphasique sur la solidification des matériaux semi-transparents.

SCHMELZ- UND ERSTARRUNGSVORGÄNGE MIT INNEREM WÄRMETRANSPORT DURCH STRAHLUNG—EIN ALLGEMEINES MODELL FÜR DEN PHASENÜBERGANG

Zusammenfassung—Es wird ein allgemeineres Modell für den Schmelz- und Erstarrungsvorgang vorgeschlagen, bei dem ein interessantes Phänomen, nämlich die Existenz eines Zweiphasengebiets, in welchem teilweiser Phasenwechsel auftreten kann, berücksichtigt wird.

Das Zweiphasengebiet entsteht durch inneres Schmelzen oder Erstarren (im Gegensatz zum Schmelzen und Erstarren an der Grenzfläche) und wird durch innere Wärmestrahlung herbeigeführt. Das vorgeschlagene Modell geht von drei physikalisch unterschiedlichen Schichten aus: der flüssigen und festen Schicht wie im klassischen Modell und einer Schicht aus einer Zweiphasenmischung (teils flüssig, teils fest), die sich zwischen den beiden anderen befindet. Es wird gezeigt, daß im Grenzfall lichtundurchlässiger Medien die Zweiphasenschicht verschwindet und sich das vorgeschlagene Modell auf den klassischen Fall reduziert. Eine entsprechende mathematische Formulierung wird für Phasenübergangsprobleme mit innerer Wärmestrahlung vorgeschlagen und dazu verwendet, einige Widersprüche in der vorhandenen Literatur aufzuklären. Schließlich werden eine exakte und zwei näherungsweise Lösungen für drei einfache Erstarrungsprobleme angegeben, um den Einfluß des Zweiphasengebiets auf die Erstarrung von halbtransparenten Materialien zu veranschaulichen.

ПЛАВЛЕНИЕ И ЗАТВЕРДЕВАНИЕ ПРИ ЛУЧИСТОМ ПЕРЕНОСЕ ТЕПЛА В ОБЪЕМЕ ВЕЩЕСТВА. ОБОБЩЕННАЯ МОДЕЛЬ ФАЗОВЫХ ПРЕВРАЩЕНИЙ

Аннотация—Предложена более общая модель плавления и затвердевания для учета интересного явления – существования двухфазной зоны, в которой могут происходить частичные фазовые превращения. Такая зона появляется в результате плавления или затвердевания внутри объема вещества (в отличие от поверхностного плавления или затвердевания), вызванного внутренним тепловым излучением. Модель включает три слоя: жидкий и твердый слои, как и в классической модели, и двухфазный смешанный слой (частично жидкость и частично твердая фаза), расположенный между ними. Показано, что в предельном случае непрозрачной среды толщина двухфазного слоя уменьшается и представленная модель сводится к классической. Предложена соответствующая формула для описания теплообмена с фазовыми изменениями при лучистом переносе тепла внутри объема вещества, которая используется для выяснения некоторых противоречий, отмечаемых в опубликованных работах. И наконец, представлены одно точное и два приближенных решения трех простых задач затвердевания, которые иллюстрируют влияние двухфазной зоны на затвердевание полупрозрачных материалов.